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A NOTE ON TRIANGULAR AUTOMORPHISMS

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Abstract. In this short note we propose a new very easy and elementary proof of the known fact that every triangular automorphism of $\mathbf{k}^n$ is the exponent of a suitably chosen locally nilpotent $\mathbf{k}$-derivation on $\mathbf{k}[x_1, \ldots, x_n]$. Two other, different proofs of this fact can be found in [**2**] and [**3**].

1. Introduction. Let $\mathbf{k}$ be a field of characteristic zero and R a $\mathbf{k}$-algebra. Recall that a $\mathbf{k}$-derivation on R is a $\mathbf{k}$-linear map $D : R \to R$ satisfying the Leibniz rule: $D(ab) = aD(b) + bD(a)$ for all $a, b \in R$. A derivation D on a ring R is called *locally nilpotent* if for every $a \in R$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. If $D : R \to R$ is a locally nilpotent $\mathbf{k}$-derivation, then the mapping $\exp D : R \to R$ given by the formula $\exp D(a) = \sum_{i=0}^{\infty} \frac{1}{i!} D^i(a)$ is a $\mathbf{k}$-automorphism of R (see e.g. [**2**] or [**4**]).

Recall also that a $\mathbf{k}$-automorphism $F : \mathbf{k}[X_1, \ldots, X_n] \to \mathbf{k}[X_1, \ldots, X_n]$ of the polynomial ring in n variables $X_1, \ldots, X_n$ over a field $\mathbf{k}$ is called triangular if $F(X_i) = X_i + f_i(X_1, \ldots, X_{i-1})$, for $i = 1, \ldots, n$. Since $\mathbf{k}$ is an infinite field, there is an isomorphism between the group of the ring $\mathbf{k}[X_1, \ldots, X_n]$ and the ring of polynomial automorphism of $\mathbf{k}^n$, given by the formula $G \mapsto G_* = (G(X_1), \ldots, G(X_n))$.

In this short note we give an easy proof of the following theorem, which has already been proved (see [**1**] and [**3**]).

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Theorem 1.1. *For all $n > 1$ and for all polynomials $f_1 \in \mathbf{k}$, $f_2 \in \mathbf{k}[X_1]$, $f_3 \in \mathbf{k}[X_1, X_2], \dots, f_n \in \mathbf{k}[X_1, \dots, X_{n-1}]$ there exists a locally nilpotent $\mathbf{k}$-derivation $D : \mathbf{k}[X_1, \dots, X_n] \to \mathbf{k}[X_1, \dots, X_n]$ such that*

$$(\exp D)_* : \left\{ \begin{array}{l} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} \mapsto \left\{ \begin{array}{l} x_1 + f_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \dots, x_{n-1}) \end{array} \right\}.$$

The proof of the above theorem can also be found in [**1**] and [**3**]. The proof given in [**1**] uses the Campbell–Hausdorff formula for $\exp D_1 \circ \exp D_2$, and the one given in [**3**] uses the notion of the logarithm of locally nilpotent map $E : \mathbf{k}^n \to \mathbf{k}^n$ (more precisely, the logarithm of $\mathrm{id}_{\mathbf{k}^n} + E$). Our proof is completely different and perhaps easier.

An easy consequence of Theorem 1.1, also already known, is the following

Corollary 1.2. *If $F : \mathbf{k}^n \to \mathbf{k}^n$ is a polynomial automorphism of the form*

$$F : \left\{ \begin{array}{l} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right\} \mapsto \left\{ \begin{array}{l} a_1 x_1 + f_1 \\ a_2 x_2 + f_2(x_1) \\ \vdots \\ a_n x_n + f_n(x_1, \dots, x_{n-1}) \end{array} \right\},$$

where $a_1, \dots, a_n \in \mathbf{k} \backslash \{0\}$, then there exists a locally nilpotent derivation $D : \mathbf{k}[x_1, \dots, x_n] \to \mathbf{k}[x_1, \dots, x_n]$ such that $F = (\exp D)_* \circ L$, where $L : \mathbf{k}^n \to \mathbf{k}^n$ is linear with the diagonal matrix determined by $a_1, \dots, a_n$.

Proof. $F \circ L^{-1}$ is of the triangular form. Following Theorem 1.1 there exists a locally nilpotent k-derivation $D : \mathbf{k}[X_1, \dots, X_n] \to \mathbf{k}[X_1, \dots, X_n]$ such that $F \circ L^{-1} = (\exp D)_*$. $\square$

2. Proof.

We start with the following lemma

Lemma 2.1. *Let R be a $\mathbf{k}$-algebra and $D : R \to R$ be a locally nilpotent $\mathbf{k}$-derivation such that for every $g \in R$ there is $\widetilde{g} \in R$ such that $g = \sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(\widetilde{g})$. For an $f \in R$ define the $\mathbf{k}$-derivation $\widetilde{D} : R[t] \to R[t]$ on the polynomial ring in one variable t over R such that $\widetilde{D}|_R = D$ and $\widetilde{D}(t) = \widetilde{f}$, where $\widetilde{f} \in R$ is such that $f = \sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(\widetilde{f})$. Then*

(1) $\widetilde{D}$ is locally nilpotent,

(2) $\exp \widetilde{D}|_R = \exp D$ and $\left(\exp \widetilde{D}\right)(t) = t + f$,

(3) for every $h \in R[t]$ there is $\widetilde{h} \in R[t]$ such that $h = \sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}^{i-1}(\widetilde{h})$.

PROOF. Assertions that $\widetilde{D}$ is locally nilpotent and $\exp \widetilde{D}|_R = \exp D$ are obvious. Moreover,

$$\begin{aligned}\left(\exp \widetilde{D}\right)(t) &= \sum_{i=0}^{\infty} \frac{1}{i!}\widetilde{D}^i(t) = \widetilde{D}^0(t) + \sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(\widetilde{D}(t)) \\ &= t + \sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(\widetilde{f}) = t + f.\end{aligned}$$

Statement (3) will be proved by induction with respect to $k = \deg_t g$. If $k = 0$, i.e., $g \in R$, then, by the assumptions, there exists an element $\widetilde{g} \in R \subset R[t]$ such that $g = \sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(\widetilde{g}) = \sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}^{i-1}(\widetilde{g})$. Now assume that (3) is true for $k \geq 0$ and consider a polynomial:

$$g = a_{k+1}t^{k+1} + a_k t^k + \ldots + a_0,$$

where $a_{k+1}, a_k, \ldots, a_0 \in R$.

By the assumptions, there is an element $b_{k+1} \in R$ such that $a_{k+1} = \sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(b_{k+1})$. Denote $\widetilde{g}_1 = b_{k+1}t^{k+1}, g_1 = \sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}^{i-1}(\widetilde{g}_1)$ and observe that

$$g_1 = \left[\sum_{i=1}^{\infty} \frac{1}{i!} D^{i-1}(b_{k+1})\right] t^{k+1} + \ldots = a_{k+1}t^{k+1} + \ldots$$

Indeed, for all $l \geq 0$ there is:

$$\begin{aligned}\widetilde{D}^l(b_{k+1}t^{k+1}) &= \sum_{i=0}^{l} \binom{l}{i} \widetilde{D}^i(b_{k+1})\widetilde{D}^{l-i}(t^{k+1}) = \sum_{i=0}^{l} \binom{l}{i} D^i(b_{k+1})\widetilde{D}^{l-i}(t^{k+1}) \\ &= D^l(b_{k+1})t^{k+1} + \sum_{i=0}^{l-1} \binom{l}{i} D^i(b_{k+1})\widetilde{D}^{l-i}(t^{k+1}).\end{aligned}$$

Since $\deg_t \widetilde{D}(h) < \deg_t h$ for each $h \in R[t]\backslash R$, we see that

$$\deg_t \widetilde{D}^j(t^{k+1}) < k + 1$$

for $j > 0$.

Thus $\deg_t(g - g_1) < \deg_t g$, and, by the induction assumption, there exists $\widetilde{g}_2 \in R[t]$ such that:

$$g - g_1 = \sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}^{i-1}(\widetilde{g}_2).$$

Putting $\widetilde{g} = \widetilde{g}_1 + \widetilde{g}_2$ we obtain

$$\sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}_2^{i-1}(\widetilde{g}) = \sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}_2^{i-1}(\widetilde{g}_1) + \sum_{i=1}^{\infty} \frac{1}{i!} \widetilde{D}_2^{i-1}(\widetilde{g}_2) = g_1 + (g - g_1) = g.$$

□

Proof of Theorem 1.1. Consider the $\mathbf{k}$-derivation $D_0 = 0$ on $\mathbf{k}$. Since $D_0^j(h) = 0$ for all $h \in \mathbf{k}$ and $j > 0$, then $\exp D_0 = \mathrm{id}_{\mathbf{k}}$ and:

$$\sum_{i=1}^{\infty} \frac{1}{i!} D_1^{i-1}(h) = h$$

for all $h \in \mathbf{k}$.

Applying Lemma 2.1 for $R = \mathbf{k}, D = D_0$ and $f = f_1$, we obtain the locally nilpotent $\mathbf{k}$-derivation $D_1 : \mathbf{k}[X_1] \to \mathbf{k}[X_1]$ such that

$$(\exp D_1)_* : \left\{ \begin{array}{c} x_1 \end{array} \right\} \mapsto \left\{ \begin{array}{c} x_1 + f_1 \end{array} \right\}$$

and that for every $h \in \mathbf{k}[X_1]$ there is $\widetilde{h} \in \mathbf{k}[X_1]$ such that $h = \sum_{i=1}^{\infty} \frac{1}{i!} D_1^{i-1}(\widetilde{h})$. Thus we can apply Lemma 2.1 for $R = \mathbf{k}[X_1], D = D_1$ and $f = f_2$. In this way we obtain the locally nilpotent $\mathbf{k}$-derivation $D_2 : \mathbf{k}[X_1, X_2] \to \mathbf{k}[X_1, X_2]$ such that

$$(\exp D_2)_* : \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} \mapsto \left\{ \begin{array}{c} x_1 + f_1 \\ x_2 + f_2(x_1) \end{array} \right\}$$

and that for every $h \in \mathbf{k}[X_1, X_2]$ there is $\widetilde{h} \in \mathbf{k}[X_1, X_2]$ with $h = \sum_{i=1}^{\infty} \frac{1}{i!} D_2^{i-1}(\widetilde{h})$.

Now it is easy to see that applying Lemma 2.1 n times, we complete the proof of Theorem 1.1 $\square$

References

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